On the structure of complete 3-manifolds with nonnegative scalar curvature

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December 6, 2011

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Abstract

In this paper we will show the following result: Let $\mathcal{N}$ be a complete (noncompact) connected orientable Riemannian three-manifold with nonnegative scalar curvature $S \geq 0$ and bounded sectional curvature $K_s \leq K$. Suposse that $\Sigma \subset \mathcal{N}$ is a complete orientable connected area-minimizing cylinder so that $\pi_1(\Sigma) \in \pi_1(\mathcal{N})$. Then $\mathcal{N}$ is locally isometric either to $\mathbb{S}^1 \times \mathbb{R}^2$ or $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}$ (with the standard product metric).

As a corollary, we will obtain: Let $\mathcal{N}$ be a complete (noncompact) connected orientable Riemannian three-manifold with nonnegative scalar curvature $S \geq 0$ and bounded sectional curvature $K_s \leq K$. Assume that $\pi_1(\mathcal{N})$ contains a subgroup which is isomorphic to the fundamental group of a compact surface of positive genus. Then, $\mathcal{N}$ is locally isometric to $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}$ (with the standard product metric).

1 Introduction

It is a fundamental question in geometry to determine the topology of a manifold $\mathcal{N}$ under assumptions on its curvature. We focus on complete noncompact Riemannian manifolds. In this line, Toponogov's Splitting Theorem [16] states that a complete Riemannian manifold $\mathcal{N}$ of nonnegative sectional curvature which contains a *straight line*, i.e., a complete minimizing geodesic, must be isometric to a product $\mathcal{N} = \tilde{\mathcal{N}} \times \mathbb{R}$. J. Cheeger and D. Gromoll [2] generalized Toponogov's Splitting Theorem under a weaker condition on the curvature of the manifold, they replaced the nonnegative sectional curvatures assumption by the nonnegativity of the Ricci curvature.

[1]The author is partially supported by Spanish MEC-FEDER Grant MTM2010-19821

In the case the sectional curvatures are strictly positive, D. Gromoll and W. Meyer [6] (see also [3]) proved that any complete noncompact manifold with positive sectional curvature is diffeomorphic to the Euclidean space. So, one could think that an analogous result holds for positive Ricci curvature, but this is false for $\dim(\mathcal{N}) \geq 4$ by the examples constructed by P. Nabonnand [11]. Hence, the question that arise is: *What is the situation in dimension three?*

So, from now on, we focus on $\dim(\mathcal{N}) = 3$. R. Schoen and S.T. Yau [15] proved that every complete noncompact three-dimensional manifold with positive Ricci curvature is diffeomorphic to the Euclidean space. When the Ricci curvature is nonnegative, M. Anderson and L. Rodríguez obtained the following result: *Let $\mathcal{N}$ be a complete, noncompact three-dimensional Riemannian manifold satisfying $\mathrm{Ric} \geq 0$ and sectional curvature bounded from above. Then, either $\mathcal{N}$ is diffeomorphic to $\mathbb{R}^3$ or the universal covering $\tilde{\mathcal{N}}$ of $\mathcal{N}$ is isometric to a product $\Sigma \times \mathbb{R}$ and all the sectional curvatures of $\mathcal{N}$ are nonnegative.* In [7], G. Liu has removed the hypothesis on the sectional curvature on the above result.

When we relax the hypothesis on the curvature to scalar curvature, R. Schoen and S.T. Yau [15, Theorem 4] proved: *Let $\mathcal{N}$ be a three-dimensional manifold. Suppose $\pi_1(\mathcal{N})$ contains a subgroup which is isomorphic to the fundamental group of a compact surface of positive genus. Then $\mathcal{N}$ cannot carry a complete metric of positive scalar curvature.* Recently, M. Micallef and V. Moraru gave a unified approach to various splitting and rigidity theorems of complete three-manifolds with scalar curvature bounded below assuming the existence of a closed, emebedded, oriented, two-sided, area-minimizing surface (see [9] for details and references therein).

Complete three-manifolds with nonnegative scalar curvature are crucial in Mathematical General Relativity, they arise as initial data sets on the time-symmetric case. In [14], R. Schoen and S.T. Yau proved the *Positive Mass Conjecture*, it states that for a nontrivial isolated physical system, the total energy, which includes contributions from both matter and gravitation is positive. In the time-symmetric case, the *Positive Mass Theorem* [13] asserts: *Let $\mathcal{N}$ be an oriented three-manifold asymptotically flat. If its scalar curvature is nonnegative, then the total mass of each end is nonnegative.* Therefore, it is a fundamental question in Mathematical General Relativity the topology of complete three-manifolds of nonnegative scalar curvature.

In this paper we focus on the case of complete, noncompact Riemannian manifolds with nonnegative scalar curvature. We prove in Section 2:

Theorem 1.1. *Let $\mathcal{N}$ be a complete (noncompact) connected orientable Riemannian three-manifold with nonnegative scalar curvature $S \geq 0$ and bounded sectional curvature $K_s \leq K$. Suposse that $\Sigma \subset \mathcal{N}$ is a complete orientable connected area-minimizing cylinder so that $\pi_1(\Sigma) \in \pi_1(\mathcal{N})$.*

Then, $\mathcal{N}$ is locally isometric to either $\mathbb{S}^1 \times \mathbb{R}^2$ or $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}$ (endowed with the standard product metric).

Recall that a surface $\Sigma \subset \mathcal{N}$ is area-minimizing if, for any domain $\Omega \subset \Sigma$ with $\overline{\Omega}$ compact, it holds

$$\mathrm{Area}(\Omega) \leq \mathrm{Area}(V),$$

where V is any oriented surface in $\mathcal{N}$ with $\partial V = \partial\Omega$.

We might expect to generalize Theorem 1.1 in two directions, we could either relax the area-minimizing hypothesis by stability, or drop the condition on the sectional curvature. In [4], the author investigated the first posibility, nevertheless if we only assume stability we need to add other hypothesis (see [4] for details and counterexamples without any other condition). Theorem 1.1 is based on the construction of a foliation by area-minimizing cylinders near Σ and, for doing so, both conditions, the area-minimizing property of Σ and boundedness of the sectional curvatures, are essential.

We use Theorem 1.1 to obtain topological obstructions on a Riemannian three-manifold carrying a complete metric with nonnegative scalar curvature. We prove the following theorem in Section 3.

Theorem 1.2. *Let $\mathcal{N}$ be a complete (noncompact) connected orientable Riemannian three-manifold with nonnegative scalar curvature $S \geq 0$ and bounded sectional curvature $K_s \leq K$. Suposse $\pi_1(\mathcal{N})$ contains a subgroup which is isomorphic to the fundamental group of a compact surface of positive genus. Then, $\mathcal{N}$ is locally isometric to $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}$ (endowed with the standard product metric).*

Theorem 1.2 is based on the fact that the topological condition on $\mathcal{N}$ implies the existence of an area-minimizing cylinder in $\mathcal{N}$. Hence, by Theorem 1.1, we obtain the result by removing the case $\mathbb{S}^1 \times \mathbb{R}^2$ using the assumption on the fundamental group of $\mathcal{N}$.

2 Splitting Theorem

In this section we will prove Theorem 1.1. The idea of Theorem 1.1's proof relies on [1] and it goes as follows:

- The existence of a complete area-minimizing cylinder $\Sigma \subset \mathcal{N}$ implies that Σ is flat, totally geodesic, the scalar curvature vanishes along Σ and it is properly embedded.
- We construct a sequence of area-minimizing sequence of surfaces with boundary on Σ and a homogically trivial loop near Σ (disjoint to Σ). This sequence will produce an area-minimizing cylinder. Actually, we construct a one parameter family. To do so, we shall see that the sequence does not vanishes and does not agree with Σ when we take the limit.
- Once we construct this one parameter family, we see that it is a foliation near Σ by totally geodesic flat cylinders so that the scalar curvature vanishes along each leaf of the foliation.
- This implies that locally, the ambient manifold is locally isometric to a product flat space. The above construction is uniform, so we can keep and going, and we can prove that the total ambient space is locally isometric to a product flat space.

Now, we detailed its proof:

Theorem1.1. *Let $\mathcal{N}$ be a complete (noncompact) connected orientable Riemannian three-manifold with nonnegative scalar curvature $S \geq 0$ and bounded sectional curvature $K_s \leq K$. Suposse that $\Sigma \subset \mathcal{N}$ is a complete orientable connected area-minimizing cylinder so that $\pi_1(\Sigma) \in \pi_1(\mathcal{N})$.*

Then, $\mathcal{N}$ is locally isometric to either $\mathbb{S}^1 \times \mathbb{R}^2$ or $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}$ (endowed with the standard product metric).

Proof. First, note that since Σ is area-minimizing, we can assume that Σ is embedded (using cut and paste argument and rounding the corners), but not necessarily proper. Moreover, from [15, Theorem 2], we know that Σ is flat, totally geodesic and the scalar curvature of $\mathcal{N}$ vanishes along Σ. Set $\mathcal{C} := \mathbb{S}^1 \times \mathbb{R}$ the flat cylinder, then we can parametrize Σ as the isometric immersion $\psi : \mathcal{C} \to \mathcal{N}$ where $\Sigma := \psi_0(\mathcal{C})$. Also, set $N_0 : \mathcal{C} \to N\Sigma$ the unit normal vector field along Σ. We will see that, under this conditions, Σ is, in fact, proper.

Claim A: *Let Σ be an area-minimizing embedded surface isometric to the flat cylinder $\mathcal{C}$ and totally geodesic. Then Σ is proper.*

Proof. Assume that Σ is not proper. Then, there exists a divergent sequence $\{p_k\} \subset \Sigma$ so that $p_k \to p_0 \in \mathcal{N}$. Set $t_k \to +\infty$ such that $p_k \in \gamma_k \subset \Sigma$, where $\gamma_k = \psi_0(\mathbb{S}^1 \times \{t_k\})$. Set $l = \mathrm{Length}(\gamma_k)$ (recall that the length of γ_k is the same for all k).

Extracting a subsequence, if necessary, we might assume that $p_k \in \mathcal{B}(p_0, l)$ for all $k \geq 1$, where $\mathcal{B}(p, r)$ is the geodesic ball in $\mathcal{N}$ centered at p of radius r. Therefore,

$$\bigcup_{k \geq 1} \gamma_k \subset \mathcal{B}(p_0, 2l).$$

Fix $k > 1$ and $\theta \in [0, 2\pi]$ and let $\beta_{k,\theta} : [0, l_{k,\theta}] \to \mathcal{B}(p_0, 2l)$ be the geodesic joining $\beta_{k,\theta}(0) = \gamma_1(\theta) = \psi_0(\theta, t_1)$ and $\beta_{k,\theta}(l_{k,\theta}) = \gamma_k(\theta) = \psi_0(\theta, t_k)$. Note that $l_{k,\theta} \leq 2l$ for all $k > 1$ and $\theta \in [0, 2\pi]$. Consider the topological annulus $\mathcal{A}_k = \bigcup_{\theta \in [0,2\pi]} (0, l_{k,\theta})$ and consider $h_k : \mathcal{A}_k \to \mathcal{N}$ given by $h_k(\theta, s) = \beta_{k,\theta}(s)$, where s is the arc-length parameter along $\beta_{k,\theta}$. Thus, the surface $\tilde{\Sigma}_k = h_k(\mathcal{A}_k)$ has boundary $\partial\tilde{\Sigma}_k = \gamma_1 \cup \gamma_k$ and surface area

$$\mathrm{Area}(\tilde{\Sigma}_k) = \int_{\mathcal{A}} 1 \, dA = \int_0^{2\pi} l_{k,\theta} \, d\theta \leq 4\pi l,$$

for all $k > 1$.

But, since Σ is area-minimizating and $\partial\Sigma(t_1, t_k) = \partial\tilde{\Sigma}$, where $\Sigma(t_1, t_k) = \bigcup_{t \in (t_1, t_k)} \psi_0(\mathbb{S}^1 \times \{t\})$, we have

$$(t_k - t_1)l = \text{Area}(\Sigma(t_1, t_k)) \leq \text{Area}(\tilde{\Sigma}_k) \leq 4\pi l,$$

for all $k > 1$, which is a contraction. Therefore, Σ is proper. $\square$

Second, we want to construct an open neighborhood of Σ by constructing a foliation of area-minimizing surfaces Σ_t, $t \in (-t_0, t_0)$ for some uniform $t_0 > 0$, so that $\Sigma_0 \equiv \Sigma$ and Σ_t is isotopic to Σ. So, let us proceed to the construction of the foliation.

Fix a point $p_0 \in \Sigma$ and set $r < 0$. Let $D(p_0, r)$ be the connected component of $\Sigma \cap \mathcal{B}(p_0, r)$ containing p_0, where $\mathcal{B}(p_0, r)$ is the geodesic ball in $\mathcal{N}$ centered at p_0 of radius $r > 0$. Denote $C(p_0, r) = \partial D(p_0, r)$

For $t \geq 0$, set

$$C(p_0, r, t) := \left\{ \exp_p(tN(p)) \ : \ p \in C(p_0, r) \right\},$$

where $\exp$ denotes the exponential map in $\mathcal{N}$.

Choose $\epsilon > 0$ small enough so that $D_\epsilon := D(p_0, \epsilon)$ bounds an embedded topological disk on Σ and $R_i > \epsilon$ a sequence of real number so that $R_i \to +\infty$ and $D_i := D(p_0, R_i)$ bounds a topological annulus on Σ with boundary $C_i := \partial D_i$ for all i.

Set $2t_0 := \min \left\{ i(p) \ : \ p \in \overline{D(p_0, 2\epsilon)} \right\}$, where $i(p)$ is the injectivity radius at $p \in \mathcal{N}$. Note that t_0 only depends on the maximum of the sectional curvatures K_s of $\mathcal{N}$ in $\mathcal{B}(p_0, 2\epsilon)$. Therefore, for all $|t| \leq t_0$, $C_\epsilon(t) := C(p_0, \epsilon, t)$ is an embedded simple loop in $\mathcal{N}$.

Claim B: *Set* $\Omega(\epsilon, R) := \{p \in \Sigma \ : \ \epsilon \leq r(p) \leq R\}$, *where* $r(p) := \text{dist}_\Sigma(p_0, 0)$. *Denote by* $\Sigma(\epsilon, R, f)$ *the surface given by*

$$\Sigma(\epsilon, R, f) := \left\{ \exp_p(f(r(p))N(p)) \ : \ p \in \Omega(\epsilon, R) \right\},$$

where $f : \mathbb{R}^+ \to \mathbb{R}$ *is given by*

$$f(r) = \begin{cases} t & r < \epsilon, \\ \frac{1}{c}\text{arccosh}(c\lambda R) - \frac{1}{c}\text{arccosh}(c\lambda r) & \epsilon \leq r \leq R, \\ 0 & r > R, \end{cases}$$

where

$$\begin{aligned} \lambda &= \cos(\sqrt{K} t_0), \\ R &= \frac{1}{c\lambda}\cosh(ct + \text{arccosh}(c\lambda\epsilon)), \\ c &= \frac{t_0}{2\epsilon^2\lambda^2}. \end{aligned}$$

Then, there exists $\epsilon_0 > 0$ (depending only on K) so that for all $0 < \epsilon < \epsilon_0$, we have

$$\mathrm{Area}(\Sigma(\epsilon, R, f)) < \mathrm{Area}(\Omega(\epsilon, R)) + \mathrm{Area}(D_\epsilon) + \mathrm{Area}(\tilde{D}(\epsilon)),$$

where $\tilde{D}(\epsilon)$ is the area-minimizing disk with boundary $C_\epsilon(t)$.

Proof of Claim B. This is a straightforward computation and we refer the reader to [1, pag. 466]. □

Set $|t| \leq t_0$ and $\epsilon < \epsilon_0$, denote by $\Sigma_i(\epsilon, t)$ the area-minimizing surface with boundary $\partial\Sigma_i(\epsilon, t) = C_\epsilon(t) \cup C_i$.

Claim C: $\Sigma_i(\epsilon, t)$ is connected for all i.

Proof of Claim C. Assume $\Sigma_i(\epsilon, t)$ is disconnected. Since $C_i = \gamma_1 \cup \gamma_2$ are two embedded closed curves generating the fundamental group of Σ, then $0 \neq [\gamma_j] \in \pi_1(\mathcal{N})$, $j = 1, 2$, and therefore they do not bound any disk in $\mathcal{N}$. Thus we should have, since $\Sigma_i(\epsilon, t)$ is area-minimizing, two components $\Sigma_i(\epsilon, t) = \Sigma_i^1 \cup \Sigma_i^2$ where $\partial\Sigma_i^1 = C_\epsilon(t)$ and $\partial\Sigma_i^2 = C_i$ and so, since Σ is area-minimizing, $\Sigma_i^2 = D_i$ and Σ_i^1 is the area-minimizing disk bounding $C_\epsilon(t)$.

But, Claim B implies that we can find a topological annulus $\Sigma(\epsilon, R_i, f)$ so that

$$\mathrm{Area}(\Sigma(\epsilon, R_i, f)) < \mathrm{Area}(\Sigma_i^1) + \mathrm{Area}(\Sigma_i^2),$$

which contradicts the fact that Σ_i is disconnected. Thus, Claim C is proved. □

We continue by proving that $\{\Sigma_i(\epsilon, t)\}$ has a convergent subsequence to a connected, area-minimizing surface $\Sigma(\epsilon, t)$ with boundary $\partial\Sigma(\epsilon, t) = C_\epsilon(t)$ for all $|t| < t_0$ and $\epsilon < \epsilon_0$.

Claim D: *For all $|t| < t_0$ and $\epsilon < \epsilon_0$, $\{\Sigma_i(\epsilon, t)\}$ has a convergent subsequence to a connected, area-minimizing surface $\Sigma(\epsilon, t)$ with boundary $\partial\Sigma(\epsilon, t) = C_\epsilon(t)$. Moreover, $\Sigma(\epsilon, t)$ is not symplyconnected.*

Proof of Claim D. We give only the idea of the proof, for details we refer the reader to [1, Pag. 463]. Since $\Sigma_i(\epsilon, t)$ is area-minimizing for all i, they verify an uniform local area and curvature estimate (see [12]). Therefore a subsequence converge (in compact subsets) to a connected area-minimizing surface $\Sigma(\epsilon, t)$. By the area estimate, the limit $\Sigma(\epsilon, t)$ is embedded. The last statement is easy to see since $\Sigma(\epsilon, t)$ is the limit of non-simplyconnected surfaces. □

Next, we keep $t \in [-t_0, t_0]$ fixed and let $\epsilon \to 0$. Arguing as above (see [1, pag. 464] for details), we obtain a limit surface S in $\mathcal{N} \setminus \{p_t\}$, where $p_t = \exp_{p_0}(tN(p_0))$, which is embedded, area-minimizing and non-simplyconnected and $\partial S = \emptyset$ in $\mathcal{N} \setminus \{p_t\}$. Now, S extend acroos p_t smoothly to a complete embedded non-simplyconnected stable surface Σ_t in $\mathcal{N}$ and area-minizing in $\mathcal{N} \setminus \{p_t\}$. Recall that, since Σ_t is not simplyconnected, Σ_t is a topological cylinder (see [5]). Thus, from [15, Theorem 2], we know that Σ_t is flat and totally geodesic and then, we can parametrize Σ_t as the isometric immersion $\psi_t : \mathcal{C} \to \mathcal{N}$ where $\Sigma_t := \psi_t(\mathcal{C})$. Also, set $N_t : \mathcal{C} \to N\Sigma_t$ the unit normal vector field along Σ_t. Moreover, by Claim A, Σ_t is proper.

Now, we shall prove:

Claim E: $\{\Sigma_t\}_{t\in(-t_0,t_0)}$ *is a* C^0-*foliation of a region of* $\mathcal{N}$ *by flat and totally geodesic cylinders. In particular, the original surface* $\Sigma_0 \equiv \Sigma$ *is a leaf of the foliation and all the surfaces are diffeomorphic.*

Proof of Claim E. Up to this point, for each $t \in (-t_0, t_0)$, we have constructed a stable minimal embedded cylinder (flat and totally geodesic) Σ_t passing throught p_t and $\Sigma_0 \equiv \Sigma$.

The first thing we shall prove is $\Sigma_{t_1} \cap \Sigma_{t_2} = \emptyset$ for $-t_0 < t_1 < t_2 < t_0$. Assume the intersection is not empty. Note that, since Σ_{t_j} pass throught p_{t_j} for $j = 1, 2$, there exists a tubular neighborhood $\mathcal{U}$ of $\mathcal{I}$ so that $\Sigma_{t_1} \cap \Sigma_{t_2} \cap \mathcal{U} = \emptyset$, here

$$\mathcal{I} = \left\{\exp_{p_0}(sN(p_0)) \,:\, s \in (-t_0, t_0)\right\}.$$

Therefore, if Σ_{t_1} intersects Σ_{t_2}, for some $\epsilon_0 > 0$ and $i_0 \in \mathbb{N}$; the sequences $\Sigma_i(\epsilon, t_1)$ and $\Sigma_i(\epsilon, t_2)$ must intersect for all $0 < \epsilon < \epsilon_0$ and $i \geq i_0$. However, since $\Sigma_i(\epsilon, t_j)$, $j = 1, 2$, are area-minimizing and $\partial\Sigma_i(\epsilon, t_1) \cap \partial\Sigma_i(\epsilon, t_2) = \emptyset$, then $\Sigma_i(\epsilon, t_1)$ and $\Sigma_i(\epsilon, t_2)$ can not intersect, reaching a contradiction.

Second, we will see that if $t_i \to \bar{t} \in (-t_0, t_0)$, then $\Sigma_{t_i} \to \Sigma_{\bar{t}}$ smoothly on compact subsets. $\{\Sigma_{t_i}\}$ are complete stable minimal surfaces, then we can extract a subsequence converging to a complete stable minimal cylinder Σ' in $\mathcal{N}$ (as we did above). Since $p_{t_i} \in \Sigma_{t_i}$, we have that $p_{\bar{t}} \in \Sigma'$. Also, since $\Sigma_{t_i} \cap \Sigma_{\bar{t}} = \emptyset$ for all i, either $\Sigma' \cap \Sigma_{\bar{t}} = \emptyset$ or $\Sigma_{\bar{t}} = \Sigma'$. But, $p_{\bar{t}} \in \Sigma'$ and $p_{\bar{t}} \in \Sigma_{\bar{t}}$, so $\Sigma' = \Sigma_{\bar{t}}$.

Thus, the one parameter family $\{\Sigma_t\}_{t\in(-t_0,t_0)}$ is continuous, and Claim E is proved. $\square$

For each $t \in (-t_0, t_0)$, the lapse function $\rho_t : \Sigma_t \to \mathbb{R}$ is defined by

$$\rho_t(p) = g\left(N_t(p), \frac{\partial}{\partial t}\psi_t(p)\right).$$

Clearly, $\rho_0(p) = 1$ for all $p \in \mathcal{C}$. Also, the lapse function satisfies the Jacobi equation

$$\Delta_t \rho_t = 0, \tag{2.1}$$

since $\psi_t(\mathcal{C})$ is a stable minimal cylinder for all $|t| < t_0$. Note that, the lapse function is not negative for all $|t| < t_0$ and therefore, by (2.1) and the Maximum Principle, either ρ_t vanishes identically or $\rho_t > 0$ for each $|t| < t_0$. So, since

$$\rho_t \to \rho_0 \equiv 1, \text{ as } t \to 0,$$

we can find a uniform constant $0 < t' < t_0$ such that $\rho_t > 0$ for all $|t| \leq t'$. This implies that ρ_t belongs to the kernel of the Jacobi operator and so, ρ_t is constant since Σ_t is parabolic (see [8]). In fact, we have deduced $\rho_t \equiv 1$ for all $t \in (-t_0, t_0)$.

Thus, since Σ_t is totally geodesic,

$$\begin{array}{rccc} Y: & \mathcal{C} \times (-t_0, t_0) & \to & \mathcal{N} \\ & (p,t) & \to & Y(p,t) := N_t(p) \end{array}$$

is parallel. Also, the flow of N_t is a unit speed geodesic flow (see [10]). Moreover, the map

$$\begin{array}{rccc} \Phi: & \mathcal{C} \times (-t_0, t_0) & \to & \mathcal{N} \\ & (p,t) & \to & \Phi(p,t) := \psi_t(p) \end{array}$$

is a local isometry onto $\mathcal{V} = \bigcup_{|t|<t_0} \Sigma_t$. Therefore, Φ is a diffeomorphism onto $\mathcal{V}$, which implies that $Y : \mathcal{C} \times (-t_0, t_0) \to \mathcal{V}$ is a globally defined unit Killing vector field. This implies that $\mathcal{V}$ is locally isometric to $\mathcal{C} \times (-t_0, t_0)$.

Up to this point, we have constructed an isometric splitting of the region $\mathcal{V} \subset \mathcal{N}$. We could continue this process if we can show that each leaf Σ_t in the foliation is actually area-minimizing in $\mathcal{N}$ (we already know that they are area-minimizing in $\mathcal{N} \setminus \{p_t\}$).

Claim F: *Σ_t is area-minimizing in $\mathcal{N}$ for all $t \in (-t_0, t_0)$.*

Proof of Claim F. To prove the claim we argue by contradiction (see [1, pag. 465]). Assume Σ_t is not area-minimizing, then there is a domain $D_t \subset \Sigma_t$ and a surface S in $\mathcal{N}$ so that $\partial D_t = \partial S$ and $\mathrm{Area}(S) < \mathrm{Area}(D_t) - \alpha$ for some $\alpha > 0$. We can join $p_t \in \Sigma_t$ to S by a smooth embedded curved γ and consider the tube generated by the points at distance $\beta > 0$ from γ. We can choose β small enough so that the surface area of this tube were less that δ, δ as small as we want to. Now, removing disks $D_1 \subset \Sigma_t$ and $D_2 \subset S$ in the interior of this tube, we obtain a surface with boundary $\partial D_1 \cup \partial D_t$ and smaller area than $D_t \setminus D_1 \subset \Sigma_t \setminus \{p_t\}$ if δ (or β) is small enough. But this contradicts that Σ_t is area-minimizing in $\mathcal{N} \setminus \{p_t\}$. Hence, Σ_t is area-minimizing in $\mathcal{N}$ for all $t \in (-t_0, t_0)$. Hence, the claim is proved. $\square$

Now, we can continue the process and we construct a foliation $\{\Sigma_t\}_{t\in\mathbb{I}}$ of properly embedded area-minimizing cylinder which are totally geodesic, flat and the scalar curvature of the ambient manifold S vanishes along Σ_t, for each $t \in \mathbb{I}$. Here, either $\mathbb{I} = \mathbb{R}$ or $\mathbb{I} = \mathbb{S}^1$.

As we did above,

$$\begin{array}{rccc} Y: & \mathcal{C}\times\mathbb{I} & \to & \mathcal{N} \\ & (p,t) & \to & Y(p,t) := N_t(p) \end{array}$$

is parallel. Also, the flow of N_t is a unit speed geodesic flow (see [10]). Moreover, the map

$$\begin{array}{rccc} \Phi: & \mathcal{C}\times\mathbb{I} & \to & \mathcal{N} \\ & (p,t) & \to & \Phi(p,t) := \psi_t(p) \end{array}$$

is a local isometry, which implies that it is a covering map. Therefore, Φ is a diffeomorphism, which implies that $Y : \mathcal{C}\times\mathbb{I} \to \mathcal{N}$ is a globally defined unit Killing vector field. This implies that $\mathcal{N}$ is locally isometric either to $\mathbb{S}^1\times\mathbb{R}^2$ or $\mathbb{S}^1\times\mathbb{S}^1\times\mathbb{R}$ (endowed with the standard product metric). □

3 Structure Theorem

Now, we can prove our structure theorem for three-manifolds with nonnegative sectional curvature. It is based on the previous theorem and the fact the the topological assumptions implies the existence of an area-minimizing cylinder.

Theorem 1.2. *Let $\mathcal{N}$ be a complete (noncompact) connected orientable Riemannian three-manifold with nonnegative scalar curvature $S \geq 0$ and bounded sectional curvatures $K_s \leq K$. Suposse $\pi_1(\mathcal{N})$ contains a subgroup which is isomorphic to the fundamental group of a compact surface of positive genus. Then, $\mathcal{N}$ is locally isometric to $\mathbb{S}^1\times\mathbb{S}^1\times\mathbb{R}$ (endowed with the standard product metric).*

Proof. The proof is based on a topological result of [15]. Under the conditions above on the fundamental group, one can construct an area-minimizing cylinder Σ in $\mathcal{N}$ (see proof of Theorem 4 in [15]). Hence, Theorem 1.1 applies and $\mathcal{N}$ should be locally isometric to $\mathbb{S}^1\times\mathbb{R}^2$ or $\mathbb{S}^1\times\mathbb{S}^1\times\mathbb{R}$ (endowed with the flat metric). However, the first possibility can not occur by the assumptions on the fundamental group, therefore $\mathcal{N}$ is locally isometric to $\mathbb{S}^1\times\mathbb{S}^1\times\mathbb{R}$ (endowed with the standard product metric). This finishes the proof the theorem. □

Concluding remark

We might expect to generalize Theorem 1.1 as in [7], that is, we might expect to remove the hypothesis on the boundedness of the sectional curvature. The key point in [7] is that any stable (compact or not) surface is totally geodesic and the Ricci curvature in the normal direction

vanishes along the surface when the ambient manifold has nonengative Ricci curvature. This is not necessary true when we only ask that the scalar curvature is nonnegative, we might have simply connected stable surfaces which are not totally geodesic.

Acknowledgements

The author wishes to express his gratitute to Peter Topping for pointing us out the references [7] and [9] in relation with this paper, while the author was presenting these results on the *Second European Young and Mobile Workshop: Geometric Analisys and Partial Differential Equations*.

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